Mathematics for Political Scientists (University of Mannheim)
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Two OLS Exercises: Solve for $\boldsymbol{\beta}$

Model: $y = X\beta + \varepsilon$ and the OLS solution is $\hat{\beta} = (X'X)^{-1}X'y$.
Goal: compute $\hat{\beta}$ step by step by *inverting the relevant matrix.*

Exercise 1 (2×2 inverse): Simple Linear Regression

Data. Intercept + one regressor ($k = 1$), with $n = 3$ observations:

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \qquad y = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}.$$

Step 1. Compute $X'X$ and $X'y$.

$$X'X = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}, \qquad X'y = \begin{bmatrix} 2+3+5 \\ 1\cdot 2 + 2\cdot 3 + 3\cdot 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 23 \end{bmatrix}.$$

Step 2. Invert the 2×2 matrix $X'X$. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
Here,

$$A = X'X = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}, \quad \det(A) = 3\cdot 14 - 6\cdot 6 = 42 - 36 = 6 \neq 0.$$

Therefore

$$(X'X)^{-1} = \frac{1}{6}\begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}.$$

Step 3. Multiply to get $\hat{\beta} = (X'X)^{-1}X'y$.

$$\hat{\beta} = \frac{1}{6}\begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 10 \\ 23 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 14\cdot 10 - 6\cdot 23 \\ -6\cdot 10 + 3\cdot 23 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 140-138 \\ -60+69 \end{bmatrix} = \begin{bmatrix} \frac{2}{6} \\ \frac{9}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{3}{2} \end{bmatrix}.$$

Answer (box it): $\boxed{\hat{\beta}_0 = \tfrac{1}{3},\ \hat{\beta}_1 = \tfrac{3}{2}.}$

Exercise 2 (3×3 inverse via cofactors/adjugate): Multiple Regression

Data. Intercept + two regressors ($k = 2$), with $n = 4$ observations:

$$X = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix}, \qquad y = \begin{bmatrix} 0 \\ 3 \\ 4 \\ 7 \end{bmatrix}.$$

Step 1. Compute $X'X$ and $X'y$.

$$X'X = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 14 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \qquad X'y = \begin{bmatrix} 14 \\ 32 \\ 4 \end{bmatrix}.$$

Step 2. Invert $X'X$ using cofactors and the adjugate. For a 3×3 matrix A, $A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$, where $\operatorname{adj}(A) = C^\top$ and $C = [C_{ij}]$ is the cofactor matrix ($C_{ij} = (-1)^{i+j} M_{ij}$, M_{ij} the minor).

Here $A = X'X = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 14 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

Minors and cofactors (compute each 2×2 determinant explicitly):

$$\begin{aligned}
M_{11} &= \det\begin{bmatrix} 14 & 2 \\ 2 & 2 \end{bmatrix} = 28 - 4 = 24 && \Rightarrow\ C_{11} = +24 \\
M_{12} &= \det\begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix} = 12 - 4 = 8 && \Rightarrow\ C_{12} = -8 \quad \text{(odd sign)} \\
M_{13} &= \det\begin{bmatrix} 6 & 14 \\ 2 & 2 \end{bmatrix} = 12 - 28 = -16 && \Rightarrow\ C_{13} = -16
\end{aligned}$$

$$\begin{aligned}
M_{21} &= \det\begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix} = 12 - 4 = 8 && \Rightarrow\ C_{21} = -8 \\
M_{22} &= \det\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} = 8 - 4 = 4 && \Rightarrow\ C_{22} = +4 \\
M_{23} &= \det\begin{bmatrix} 4 & 6 \\ 2 & 2 \end{bmatrix} = 8 - 12 = -4 && \Rightarrow\ C_{23} = +4 \quad \text{(odd sign)}
\end{aligned}$$

$$\begin{aligned}
M_{31} &= \det\begin{bmatrix} 6 & 2 \\ 14 & 2 \end{bmatrix} = 12 - 28 = -16 && \Rightarrow\ C_{31} = -16 \\
M_{32} &= \det\begin{bmatrix} 4 & 2 \\ 6 & 2 \end{bmatrix} = 8 - 12 = -4 && \Rightarrow\ C_{32} = +4 \\
M_{33} &= \det\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} = 56 - 36 = 20 && \Rightarrow\ C_{33} = 20
\end{aligned}$$

Thus the cofactor matrix and the adjugate are

$$C = \begin{bmatrix} 24 & -8 & -16 \\ -8 & 4 & 4 \\ -16 & 4 & 20 \end{bmatrix}, \qquad \operatorname{adj}(A) = C^\top = \begin{bmatrix} 24 & -8 & -16 \\ -8 & 4 & 4 \\ -16 & 4 & 20 \end{bmatrix}.$$

Determinant: $\det(A) = 16$ (you may verify, e.g. by expansion).

Therefore,

$$(X'X)^{-1} = \frac{1}{16} \begin{bmatrix} 24 & -8 & -16 \\ -8 & 4 & 4 \\ -16 & 4 & 20 \end{bmatrix}.$$

Step 3. Multiply to get $\hat{\beta} = (X'X)^{-1}X'y$.

$$\operatorname{adj}(A)\,X'y = \begin{bmatrix} 24 & -8 & -16 \\ -8 & 4 & 4 \\ -16 & 4 & 20 \end{bmatrix} \begin{bmatrix} 14 \\ 32 \\ 4 \end{bmatrix} = \begin{bmatrix} 16 \\ 32 \\ -16 \end{bmatrix}, \qquad \Rightarrow \quad \hat{\beta} = \frac{1}{16} \begin{bmatrix} 16 \\ 32 \\ -16 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Answer: $\boxed{\hat{\beta}_0 = 1,\ \hat{\beta}_1 = 2,\ \hat{\beta}_2 = -1.}$

Side note: 2×2 inverse.
If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\det(\mathbf{A}) = ad - bc \neq 0$, then

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Side note: Adjugate method for 3×3.
For $A = [a_{ij}]$, compute each minor M_{ij}, then $C_{ij} = (-1)^{i+j} M_{ij}$, form $C = [C_{ij}]$, take $\operatorname{adj}(A) = C^{\top}$, and use $A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$.