Math Refresher - Extra Exercise
Optimization Example from Game Theory

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Nash Equilibria With Continuous Strategies

Two firms 1 and 2 seek to influence policy implementation by a bureaucrat B. The final policy outcome is a function of the money they spend to please the bureaucrat (some sort of corruption). Firm 1's most preferred policy is $x_1 = 1$ and firm 2 's most preferred policy is $x_2 = 0$. The bureaucrat's default policy that she would implement without being bribed is $x_B = \frac{1}{2}$. Both firms simultaneously choose an amount of money, $s_i \in [0, 1]$. The final policy is

$$x = x_B + s_1 - s_2$$

Firm 2 has fewer resources and is therefore more negatively affected by increasing amounts of money spent. The utility functions of the firms are given by

$$u_1(s_1, s_2) = -(x - x_1)^2 - s_1$$
$$u_2(s_1, s_2) = -(x - x_2)^2 - 2s_2^2$$

(a) Write down the first and second order conditions. (b) Solve for pure strategy Nash equilibria.

Solution I

First we write up the full utility functions:

$$
\begin{aligned}
u_1(s_1, s_2) &= -\left(\frac{1}{2} + s_1 - s_2 - 1\right)^2 - s_1 \\
&= -\left(-\frac{1}{2} + s_1 - s_2\right)^2 - s_1 \\
u_2(s_1, s_2) &= -\left(\frac{1}{2} + s_1 - s_2 - 0\right)^2 - 2s_2^2 \\
&= -\left(\frac{1}{2} + s_1 - s_2\right)^2 - 2s_2^2
\end{aligned}
$$

Solution II

Then we take the first derivatives:

$$
\begin{aligned}
\frac{\partial u_1\left(s_1, s_2\right)}{\partial s_1} &= -2\left(-\frac{1}{2} + s_1 - s_2\right) \cdot 1 - 1 \\
&= 1 - 2s_1 + 2s_2 - 1 \\
&= -2s_1 + 2s_2 \\
\frac{\partial u_2\left(s_1, s_2\right)}{\partial s_2} &= -2\left(\frac{1}{2} + s_1 - s_2\right) \cdot (-1) - 4s_2 \\
&= 2\left(\frac{1}{2} + s_1 - s_2\right) - 4s_2 \\
&= 1 + 2s_1 - 2s_2 - 4s_2 \\
&= 1 + 2s_1 - 6s_2
\end{aligned}
$$

Solution III

Now, we write down FOC :

$$\frac{\partial u_1(s_1, s_2)}{\partial s_1} \stackrel{!}{=} 0$$
$$-2\hat{s}_1 + 2s_2 \stackrel{!}{=} 0$$
$$\hat{s}_1 = s_2$$
$$\frac{\partial u_2(s_1, s_2)}{\partial s_2} \stackrel{!}{=} 0$$
$$1 + 2s_1 - 6\hat{s}_2 \stackrel{!}{=} 0$$
$$\hat{s}_2 = \frac{1}{6} + \frac{1}{3}s_1$$

Now, we write down the SOC:

$$\frac{\partial\partial u_1(s_1, s_2)}{\partial s_1 \partial s_1} = -2 < 0 \rightarrow \text{ Maximum}$$
$$\frac{\partial\partial u_2(s_1, s_2)}{\partial s_2 \partial s_2} = -6 < 0 \rightarrow \text{ Maximum}$$

Solution IV

Solution b: To solve the game for Nash Equilibrium means that one player has to maximize his payoff given the strategy of the other player. We insert the FOC equations and solve for $\hat{s}_2$:

$$\begin{aligned} \hat{s}_2 &= \frac{1}{6} + \frac{1}{3}\hat{s}_2 \\ \frac{2}{3}\hat{s}_2 &= \frac{1}{6} \\ \hat{s}_2 &= \frac{1}{4} \end{aligned}$$

Now, working with the previous equations gives us the strategy for the first player:

$$\hat{s}_1 = \frac{1}{4}$$

The SOC tell us that this is a maximum and both strategies are in the strategy space. The Nash equilibrium is given by
$NE = \left\{ \left(\frac{1}{4}, \frac{1}{4} \right) \right\}$